THE NATURE OF APPLIED MATHEMATICS: REMARKS ON FIELD'S VIEW

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Abstract

In this paper I raise some objections to Field's characterization of applied mathematics, showing, by means of three examples, that it is too restrictive. While doing so, I articulate a different and wider account of applicability. I conclude with an argument supporting its compatibility with an anti-realistic view on the existence of mathematical entities.

Introduction

The problem of applicability has always been more or less implicitly present in current debates in the philosophy of mathematics, especially through the discussion of indispensability arguments. Only in relatively recent times, however, (see especially Steiner (1995, 2005)) has it begun to be considered a philosophically relevant one in its own right: in particular, it has been felt necessary to *account for* the applicability of mathematical theories, i.e. to explain how they can model empirical phenomena (or be used to describe them: in this connection see again Steiner (1995, p.137)).

I embrace this perspective in this paper and apply it to a very influential book in the philosophy of mathematics, namely Field (1980), because it has never been discussed in depth as a study of applicability, despite the fact that one of its aims is precisely that of accounting for it (Field 1980, p.6).

My main thesis is that Field's view, although containing important insights into the nature of applied mathematics, severely restricts its role and does so in a way that can be shown to be unrealistic in the face of the structure of concrete applications.

To clarify this claim, some preliminary terminology and a few observations are needed. Firstly, in the context of applied mathematics, I distinguish between qualitative theories, which directly describe possibly idealized empirical domains, and the mathematical theories that are applied to them, which do not and may contain primitives or defined concepts that have no empirical counterpart (e.g. in affine geometry, taken as the geometry of space-time, no operation is defined on points, whereas in its analytical version the addition of coordinates is defined). Here 'qualitative' may be understood as synonymous with 'nominalistically acceptable' in Field's sense[1].

With this distinction in place, Field (1980) can be said to maintain that mathematical theories enter applications as:

a) Devices for shortening proofs, i.e. carrying out deductions from qualitative theories more simply and quickly (e.g. within a numerical framework).
b) Languages wherein qualitative theories may be formulated[2]. Here Field claims that platonistic scientific theories (ones formulated in mathematical terms) can always be replaced by qualitative counterparts, in which the use of mathematical terms is avoided.

Both (a) and (b) are based on the idea that mathematics plays a role in applications, which is subordinated to the availability of a qualitative theory. For in (a) deductions already possible within a qualitative theory can simply be translated and more expeditiously carried out within a mathematical, typically numerical context. On the other hand, (b), given Field's claim, implies that a mathematically formulated theory can be accepted as empirically meaningful only insofar as it is possible to isolate its qualitative content.

1 A qualitative theory is nominalistically acceptable because it has idealized empirical interpretations and thus may count as an 'empirical' theory. The same holds for all nominalistically acceptable theories considered in Field (1980) (even though some of them, as affine geometry, have mathematical interpretations, e.g. a numerical one: indeed this is needed, according to Field, for the applicability of mathematics to be possible, insofar as he bases it on the provability of representation theorems).

2 Think of classical mechanics, where e.g. Newton's second law of dynamics is formulated as an algebraic equation between numbers (if we take the components of the vector equation).

While I recognize the fundamental importance of qualitative theories to illustrate the way mathematics is related to a qualitative content, I also believe the role played by mathematical theories in applications should not be restricted to (a) and (b). In many cases, the use of mathematical theories is not conceptually subordinated to the availability of a qualitative theory but, on the contrary, directs and organizes its emergence.

In particular, mathematics is used as:

1) A guide to the qualitative characterization of an empirical domain;
2) A means of analysis of the structure of empirical data;
3) A source of (metatheoretical) explanations for the connections between a qualitative theory and a mathematical one.

As can be seen, (1) to (3) differ from (a) and (b). It remains to show that they correspond to the actual employment of mathematics in applicative contexts. The next sections are devoted to doing this and articulating a characterization of applied mathematics richer and stronger than Field's and based on these examples.
This raises an ontological issue, since Field describes applicability in such a way as to make it consistent with its nominalistic project while, on the other hand, it is not obvious whether a different and stronger account of applicability like the one I propose can be retained together with mathematical antirealism. In the last section of the paper I show that it can.

The nature of physical magnitudes

My first illustration of how mathematical theories can guide the development of qualitative theories in sense (1) of section 1 is extensive measurement, i.e. the measurement of additive magnitudes (lengths, masses etc.). This is often formalized by means of an axiomatic theory, which provides a qualitative characterization of magnitudes, and entails a metatheorem to the effect that any model of the theory (any system of magnitudes) can be embedded into the additive, positive reals (arithmetical addition on the reals is just the numerical interpretation of an additive operation on magnitudes like, e.g., the juxtaposition of rods in the case of length measurement). This means that all systems of extensive magnitudes modelling the axioms have a measurement scale on the real numbers.

The interesting fact concerning this example is that, if one allows for the possibility of infinitely large domains of magnitudes, a proof that a real measurement scale exists requires the assumption of a condition of finite comparability between magnitudes, called Archimedes' axiom[3]. It is this axiom whi<h governs the process of approximation to the 'true' numerical measure of a magnitude and ensures that it is a uniquely determined real number. The classical theory of magnitudes, as given in e.g. Huntington (1902), uses, instead of Archimedes' axiom, the condition of Dedekind completeness[4], which implies it and forces the structure of a system of magnitudes to be structurally identical to the linear continuum of the reals (something similar happens to lines in the affine geometry used by Field to describe space-time in Field (1980)). In principle this result is delivered by the Archimedean axiom alone because, in presence of the other assumptions of the theory of extensive measurement, it ensures the existence of a unique Dedekind completion for any model of this theory (i.e. a Dedekind complete extension of the model).

The crucial point is that the neither Dedekind completeness nor Archimedes' axiom is assumed on the basis of the observable behaviour of magnitudes. The reason is that Dedekind completeness is untestable, being a statement about infinite sequences of magnitudes, which would require an (unfeasible) infinitely precise measurement

3 This says that if, according to some empirical ordering, magnitude a is smaller than b, there exists, for some n, na, the result of empirically adding together n copies of a, such that na is greater than b.

4 This condition states, for real numbers, that any increasing sequence of reals which is bounded above (i.e. such that there is a fixed real greater than any element of the sequence) has a limit, i.e. a number that bounds it above and is the least number to do so. It is easy to see how the same assumption can be rephrased for extensive magnitudes.

procedure to be checked; on the other hand, as long as experimental practice is restricted to dealing with finite domains of magnitudes, Archimedes' axiom can be dispensed with in the proof of the existence of a numerical scale.

Because of this, Dedekind completeness and Archimedes' axiom are *theoretical* assumptions concerning magnitudes. Strictly speaking, they cannot be considered extrapolations from experiment. In particular Archimedes' axiom imposes a condition on infinitely large domains of magnitudes (or, equivalently, their asymptotic behaviour) which is automatically satisfied by sufficiently structured domains when their size is finite.

To see a way in which finiteness makes the Archimedean condition dispensable[5], consider the elementary case of mass measurement performed with an equal arm balance. In this context an ordering on masses is defined by looking at whether the balance is or is not in equilibrium (it may be assumed that the balance be infinitely precise, since imprecision doesn't bear on the point under discussion): thus, given objects x and y, $x >_m y$ means that, when x and y are put on the distinct pans of the balance, x descends while y ascends. A weak ordering on masses (i.e. a transitive and connected binary relation) can be defined by similar observations. In addition, it is possible to put several objects on a single pan of the balance: if they are called $x_1 \ldots x_n$, then the sum $x_1 +_m \ldots +_m x_n$ is their physical addition, whose mass equals the mass of any object which equipoises $x_1 \ldots x_n$ when they are all on one of the balance's pans. Order and addition on masses defined on a domain M of physical objects determine a structure $\mathbf{M} = \langle M, \geq_m, +_m \rangle$ abstractly describing an experimental measurement setting: the experimental procedures applied to this setting always give rise, if actually performable, to finitely many comparisons between concatenations of masses.

5 In this connection also see Suppes (1969, p.4–8).

Given a first order language for **M**, this finite number of comparisons can be denoted by a finite set of atomic formulas, which might be read, if a scale of measurement existed, as linear inequalities. The possibility of finding real measures for the elements of M (supposing it to be finite) is then reduced to the existence of a solution of a system of linear inequalities in the positive reals (this also means that $\geq_m$ and $+_m$ can be respectively interpreted as order and addition on the positive reals). This way of setting up a scale of measurement does not require an Archimedean axiom. The reason can be briefly illustrated as follows: consider a set of axioms A for extensive measurement, e.g. the one in Huntington (1902) mentioned above, and call A′ the axiom system obtained from A by removing completeness (so that no Archimedean condition can be proved from the remaining axioms) and adding the commutativity of the binary operation $+_m$.

It has been shown in Adams et al (1970)[6] that the existence of a structure-preserving mapping from **M** into the positive, additive reals (i.e. the existence of a real solution for a system of linear inequalities generated by a structure of the type of **M**) is equivalent to **M** satisfying A′[7].

Thus, whenever we want to measure a finite extensive structure, we can make use of an algorithm, providing the solution of a system of linear inequalities, which does not rely on the usual technique of successive approximations based on Archimedes' axiom.

This result shows that, while the real, Archimedean continuum may provide an important heuristic model to study extensive measurement, its choice is by no means

6 The original result, for which cf. Adams et al (1970, p.397–398), is formulated in a different manner and for a different axiom system (namely, a modified version of Suppes (1951)). However A' entails the statements of that axiom system and, on the basis of this, my statement of the result is essentially the same as the original one.

7 Since the existence of a structure-preserving mapping as the one above also entails that all scales of measurement for **M** are related by a positive constant factor, this result itself is independent, for finite extensive structures, of Archimedes' axiom.

forced upon us by experiment. It therefore appears plausible to look at certain non Archimedean structures extending the reals (i.e. structures which may be viewed as enlargements of the reals, containing infinitely large or small numbers, which violate Archimedes' axiom) and examine the possibility of whether they may provide an alternative numerical model guiding the abstract conceptualization of extensive magnitudes.

It turns out that, following this approach, systems of magnitudes which are structurally different from the reals but admit a measurement scale on them can be described. Their particular qualitative characterization is suggested by a purely numerical fact, namely the relationship between the real numbers (more precisely, a structure like the real field or, possibly, the full real structure) and their non Archimedean elementary extensions, which may be obtained by means of the compactness theorem of first order logic. An application of the latter makes it possible to determine, given a suitable real structure, an extension thereof containing infinitesimals[8]. If a sufficiently rich real structure is considered (e.g. the ordered field structure, including addition and multiplication) it is possible to 'add' to its domain infinitely large and small numbers (i.e. numbers that are respectively greater than any positive integer or smaller than its reciprocal in the ordering extending '>' on the reals). In particular real numbers may be surrounded by other numbers that are infinitely close to them.

While (i) any two real numbers have a finite difference, and thus a sequence tending to one of them as its limit must at some point finitely diverge from the other, (ii) in the elementary extensions of the reals there may be sequences that have only an infinitesimal divergence and thus end up being infinitely close to the same real limit.

8 A proof of the fact that elementary extensions of the full real structure contain infinitesimals (but the ordered field structure is sufficient to achieve it: as a matter of fact, because the field structure imposes the existence of multiplicative inverses, the existence of infinitesimals entails that of infinitely large numbers) may be found in Narens (1974, p.378– 379). An alternative, algebraic construction of elementary extensions of the reals using ultraproducts is very clearly described in Goldblatt (1998, p.23–27).

As long as the reals are used as a privileged model for the qualitative characterization of extensive magnitudes, the latter's behaviour conforms to (i): as a result, magnitudes can be identified with uniquely determined physical states, corresponding to the limits of certain sequences of approximations. On the other hand, (ii) makes it possible to think of magnitudes not as uniquely determined physical states, but as ones which are subjected to infinitesimal oscillations (it is sometimes assumed, in physics, that 'negligible', infinitesimal variations of magnitudes are possible: for example see Segel (1991)). It follows that the mathematical relationship between the reals and their elementary extensions, as briefly outlined by the contrast between (i) and (ii), suggests two alternative characterizations of the qualitative structure of additive magnitudes.

At the same time, it is noteworthy that (the possibility of doing so is rigorously shown in Narens 1974), if one thinks of magnitudes as oscillating within infinitesimal regions, it is still possible (as long as no infinitely large or infinitely small ones arise) to scale them on the reals, for infinitely close magnitudes can just be assigned the same real measure, as if their infinitesimal difference were negligible.

The results of ordinary measurement practice are compatible with a qualitative understanding of magnitudes which diverges from the classical one. But such a qualitative understanding has been suggested by the mathematical relationship between two numerical structures. This means, therefore, that numerical facts can be used to modify the intrinsic characterization of physical entities like lengths or masses or, equivalently, to integrate experimental data into different abstract frameworks.

The chronological order of deposits

Point (2) of section 1 qualifies mathematics as a means of data analysis. To justify this remark and explain exactly how (2) differs from Field's account of applicability, I will use an example taken from archaeology. It is often the case that archaeologists need to solve what are called seriation problems. A seriation is the chronological ordering of different deposits within one or more distinct sites: this problem is relevant to the present discussion because it often happens that certain seriations are not known nor can be determined by chemical or physical tests. In this situation one needs to find a chronological ordering of deposits, starting only from information concerning the artefacts they contain. Because of this, a representational strategy like the one used in Field (1980) to describe applicability, is not available nor pertinent to the problem at hand. The reason why this is so is twofold: firstly, we cannot numerically label the deposits to obtain a numerical chronology, because we haven't got enough information to assign the appropriate labels; secondly, we primarily wish to determine a qualitative chronological order, rather than giving the numerical description of one.

Now, mathematics enters this problem not as a way of formulating a theory of order for deposits or as a way of expediting proofs concerning qualitative ordered structures, but as a way of processing archaeological data to generate a chronology. Mathematics works as an instrument guiding the identification of a qualitative structure. The details of how the seriation problem may be solved can be found in Shuchat (1984): here I restrict myself to outlining the mathematical strategy adopted, in order to clarify the nature of its application in this context. The seriation problem is in essence solved by giving a suitable mathematical presentation of the available data and, subsequently, by working on the abstract configuration of data thus obtained by means of relevant mathematical concepts and theorems. The data the archaeologist has at her disposal concern the number of deposits she wants to order and a classification into types of the artefacts occurring in those deposits[9]. Deposits and types are arranged into an incidence matrix, whose

9 Deposits are identified with, roughly, 'points' in time at which they were formed, while types are identified with period of time during which they were in use.

entries a_{ij} are only 1 or 0: in particular, $a_{ij} = 1$ if deposit i contains type j while $a_{ij} = 0$ otherwise.

Under the assumption (h) that a given type be present in all deposits corresponding to the period through which it existed, it follows that the chronological ordering of the deposits, i.e. the chronological ordering of the rows of the incidence matrix $[a_{ij}]$, must give rise to a matrix whose columns do not contain any zero-entries lying between the 1-entries (otherwise (h) would be contradicted). A matrix satisfying this property is called a *Petrie matrix*, whereas a matrix M such that a permutation P of its rows generates a Petrie matrix is called a *pre-Petrie* matrix.

The problem of seriation is thus reduced to the mathematical problem of finding the necessary and sufficient conditions under which an incidence matrix is a pre-Petrie matrix: if it is not (in case, that is, (h) does not hold), the goodness of fit of a seriation can still be estimated, by finding a measure of how close an incidence matrix is to a pre-Petrie form.

Note, again, that the mathematical solution to the seriation problem is not based on the mathematical deduction of facts which could also be deduced from an associated qualitative theory: here we are not working with a general theory of finite orderings, but with a type of data configuration from which an ordering has to be extracted.

Once the seriation problem is formulated in terms of matrices, its solution comes from establishing a correlation between two mathematical theories, namely matrix algebra and the theory of networks (undirected graphs). This is because, multiplying M by its transpose[10], one obtains a symmetric matrix S, such that s_{ij} = (number of types common to deposits i and j). S is called a *similarity* matrix

10 The transpose of A is obtained from A by interchanging its rows and columns.

because its entries tell us how many types occur simultaneously in any two deposits.

If there are n types, then $d(i, j) = n - s_{ij}$ tells us how many types occur in one but not in the other deposit, i.e. gives us a measure of dissimilarity between them. This shows that there is an explicit relationship between the data configurations described by M and S and the qualitative content of the seriation problem: nonetheless, the solution to the problem – this is the crucial point – follows from a purely abstract analysis of the data configuration involved (none of the operations performed on it has a qualitative interpretation).

Using the dissimilarity function d, we can associate to an incidence matrix M a *path* through its m rows (given by the order in which the rows follow one another), to which the length $\Sigma d(i, i + 1)$ is associated (i ranges over the m rows of M[11]). To any permutation of the rows of M there corresponds a path, whose length can be computed[12]. It is intuitively clear that, if the length of the path turns out to be minimal, we have minimized the dissimilarities between adjacent rows, thereby obtaining a chronological ordering. In particular it can be shown that a lower bound for the paths' length exists, which is attained if and only if M is a pre-Petrie matrix. If on the other hand M is not pre-Petrie, it is possible to work on the difference between the length of its associated paths and their lower bound: minimizing it leads to a 'best' approximation to a pre-Petrie matrix and, consequently, to a candidate solution for the seriation problem.

This method can be refined in several ways, for instance by using stochastic matrices instead of incidence ones (whose entries are the probabilities of finding

11 Usually one adds an arbitrary 0-row to get the domain of i right and, when i = m, identifies m+1 with the 0-row.

12 This is actually a simplification of the original mathematical treatment, which involves Hamiltonian paths. I do not talk about them here because I'm only interested in presenting the mathematical strategy in an intuitive, informal fashion.

a certain type in a certain deposit) and provide a more realistic framework to describe chronologies. In any event, the mathematical procedure adopted to find a chronological ordering entirely relies on an examination of the mathematical properties of a data configuration. What is gathered from it is the linear arrangement of a qualitative structure.

The continuous distribution of the gravitational potential

To illustrate (3) of section 1, I consider a problem in Field (1980) and the way it is implicitly solved therein. The problem is to find the qualitative formulation of an analytical property of functions (thus a mathematical property). More explicitly, the ordinary treatment of the classical theory of gravitation (developed in terms of fields), that Field nominalises, involves the reference to a function f measuring the gravitational potential at any point in space and which may be defined through time as well. This is a numerical function from ordered quadruples of reals (the coordinates of space-time points) into real numbers (the values of the potential at those points).

The continuity of f can be defined by means of the usual Weierstrass ε-δ condition, stating that, for any point x in space-time, whenever a positive real ε is chosen, it is possible to find an open ball of centre x and radius δ, such that all points of space-time in it have a value of potential that differs by less than ε from f(x).

Because this definition involves a reference to the concept of distance (the radius of an open ball), it is not, as stated, expressible qualitatively within the framework Field adopts, for both the axiomatic geometry of space-time and the qualitative theory of the potential he uses lack, among their primitives, a notion of distance or congruence: their primitives are only order relations.

In order to express continuity in this context one has to make it independent of distance: that this is intuitively possible follows from the fact that the metrical definition given above only involves, in essence, the idea of the closeness together of certain points. Closeness can be described in terms of order alone, by talking about arbitrary (and thus also arbitrarily small) intervals or regions, which may be defined in terms of order alone.

What Field does is to resort to a more general notion of continuity, which is formulated in the context of a mathematical theory, that of topological spaces. Here it can be proved that, under certain conditions (the so called first axiom of countability[13]), the metrical, analytical definition of continuity is equivalent to one formulated in terms of open sets, which is nonmetrical in the sense that it does not require a notion of distance in its formulation (a function f from a topological space S into a topological space T is continuous when, for any open set X of T, $f^{-1}(X)$ is an open set of S).

Such nonmetrical definition can be formulated using only the primitives of affine space-time geometry and the (affine) theory of potential discussed by Field: thus, exploiting topological continuity, he can find a qualitative characterization of the mathematical property of continuity.

Topology works here as an abstract mathematical theory bridging a qualitative and a numerical theory. Semantically, any model of geometry or its real counterpart can be endowed with a topology and the availability of a topological notion of continuity makes it possible to interpret it within different models and, as a consequence, to connect different theories.

This semantic fact can be also understood as a relationship between the languages of the three theories involved, for the general definition of continuity in topological terms can be translated within the other theories by means of their primitives. At the same time, the assurance that this new notion of continuity is strong enough

13 For the technical details see Pfanzagl (1968, Ch. 2).

to mirror the classical one is provided by a theorem of topology and therefore established deductively.

These remarks show that the importance of abstract mathematical theories in applications can be reduced to their deductive content, leading to the introduction of certain important logical connections between the relevant concepts. In other words, it is not their possible reference to abstract entities which makes the mathematics useful or conceptually crucial. I will get back to this point in the concluding section, in order to explain why my view on the applicability of mathematics, although different from Field's, remains compatible with an anti-realistic position concerning the existence of mathematical entities.

My main point for now is that, from the example above, one can see how mathematical theories work in applications as metatheoretical explanations for the connections between a qualitative theory and a mathematical one. The previous discussion has shown that the assumption of continuity, ordinarily made in the analytical treatment of the gravitational potential, is physically meaningful, because it can directly be described as a property of the qualitative variation of a magnitude in space-time.

This result, which is obtained at a metatheoretical level (because it involves the interaction of several theories and the interpretation of concepts from one theory into another), explains precisely why it makes sense to use the numerical property of continuity to describe an ordered physical geometry and a quantity varying over it: without it, the desired qualitative theory (one inducing continuity on the analytical representation of the potential) could not be formulated and thus we have a situation where mathematics is used to guide the formulation of a qualitative theory.

Conclusion: Applicability and realism

I take it that I have shown in the foregoing discussion how mathematical theories can enter applications in a way that does not correspond to any of those highlighted by Field ((a) and (b) of section 1). Now I wish to emphasize the common aspects of the examples I have given and formulate precisely the reason why they are compatible with anti-realism about mathematical entities. By doing so, I intend to make apparent that a significantly richer role than the one identified by Field may be ascribed to the applicative use of mathematics, while this does not necessarily require to embrace some form of mathematical realism.

As already remarked, all the examples I have proposed describe a way in which a mathematical theory aids the characterization of a qualitative theory or a qualitative structure. The former case was illustrated in my last example, where it was possible to identify, by means of topology, a qualitative axiom corresponding to the analytical condition of continuity of the gravitational potential. The latter case (characterizing a qualitative structure) was described in my second example on archaeology, where the problem at hand was to identify a qualitative order. Finally, my first example involves, in a sense, both the aspects occurring in the other two, for it illustrates a change in the qualitative characterization of additive magnitudes but also an alternative basis for their numerical representation on the reals.

Thus all the examples I have proposed highlight the fact that mathematical theories (non standard analysis, the theory of networks and the theory of real vector spaces) are not necessarily subordinated to corresponding qualitative theories in applications, in the sense that they can legitimately be used only in presence of an independent and fully informative qualitative translation thereof.

In the case of extensive measurement a new qualitative theory is suggested *through* mathematics and the general strategy of Field (1980) is, after all, to take the standard analytical treatment of classical gravitation theory and look at the kind of qualitative structure it induces (i.e. the kind of axioms a physical domain must satisfy in order for them to be analytically representable in the standard way). Thus, it is apparent that mathematics plays a heuristic role in the formulation of qualitative theories. This resembles (b), but it is crucial that in (b) we only have a given mathematically formulated theory which we then must try to nominalize, whereas e.g. in my first example we can clearly already have a qualitative theory and then modify it on the basis of certain mathematical results (the relation between the reals and their elementary extensions). The same is also true of my last example.

Finally, the discussion of seriation shows how one can reconstruct qualitative information simply by giving a mathematical presentation of a set of data and working on it within the context of a mathematical theory. Again, it is obvious that this process is attached to the original qualitative setting (indeed it must be), but its relevant feature is that it acts as a form of abstract data analysis and not as a mathematical translation of qualitative facts or as a deduction from a numerical representation thereof.

Because of these remarks, it follows that Field's characterization of applied mathematics is very restrictive and should be enlarged, if it were to cover a sufficiently wide range of situations and thus provide as comprehensive as possible an account of applicability. One worry might arise in this context, due to the possibility that the acceptance of the wider account of applicability I have outlined may imply a realistic commitment to mathematical entities, which obviously the nominalist wishes to avoid.

However, this worry can be dissipated by showing that the uses of mathematical theories I considered are compatible with mathematical antirealism. The reason why it is so has already been mentioned in the previous section.

It is, essentially, that mathematical theories are fruitful as guides in the construction or analysis of qualitative theories or structures because of the logical analysis they provide of these theories and structures, which, in turn, is articulated deductively by their theorems. It is not their referring to abstract objects to be actually used, but rather their proving certain results which direct the way qualitative settings may be dealt with. At the same time, these results are motivated by the nature of the applicative problems at hand.

For instance, in my first example the relationship between the real numbers and their elementary extensions is based on viewing the former as measures or evaluations of magnitudes, rather than purely mathematical entities (there is no space here to elaborate upon this remark: let it suffice to say that numerical measures can essentially be constructed as indices encoding experimental information coming from observational reports, as shown in Niederée (1992)). The application of the compactness theorem illustrates how these measures can describe two different idealized processes of empirical approximation, involving limits or infinitesimal differences. Thus, the purely algebraic relation between the reals and their non Archimedean extensions takes its motivation from the need to characterize the asymptotic behaviour of empirical approximations. This is confirmed by the fact that Narens (1974) contains a theorem showing the embeddability of extensive magnitudes structures for which Archimedes' axiom is not assumed into elementary extensions of the reals, a result which parallels the embeddability of the Archimedean structures into the reals. Because the concept of representation is involved in both cases, a representational treatment of measurement in the style of Field (1980) goes through: however, it is clear that such treatment is subordinated to the analysis of the relation between the numerical representing structures (the reals and their extensions), which is used as a heuristic to reconceptualize the notions of magnitude and of measurement scale.

In the example from archaeology, on the other hand, the relevant theorems are those of matrix algebra and the ones giving bounds for certain networks (the paths associated to the incidence matrices and their permutations), motivated by the need to analyze data configurations. Finally, in the case of gravitation theory the key result is a theorem of topology, establishing the equal strength of two definitions of continuity and motivated by the need to isolate the essential features of an analytical concept of quantitative variation, in order to make it expressible within a non-numerical setting.

It is through the above theorems that we can better understand how to deal with qualitative settings: as long as the focus is on them, i.e. on deductively obtained correlations motivated by ultimately empirical needs, it is not necessary to regard mathematics as carrying with itself an inescapable ontological commitment, because it is used essentially to direct model construction through proofs or to reflect upon the features of empirical models on the basis of data coming from them.

As a result, it is possible to keep together an anti-realistic view similar to Field's with the account of applied mathematics I have expounded here.

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